Co/homology operations from E_∞ operads

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We have been talking about Steenrod squares and seen some constructions. Seperately to this Diarmuid has wanted me to look at Dyer-Lashoff operations on the homology of infinite loop spaces. So I wanted to see how these things can be related. Moreover we will hopefully see some better intuition for where Steenrod operations are comming from.

1 Whats an Operad

Following [MSS07]. Operads are one of these annoying but powerful and maybe beautiful things in mathematics that have about 10 different equivalent definitions. We will be vague on some of the categorical background and state Mays original definition abstracted to a symmetric monoidal category.

1.1 Symmetric monoidal categories

Recall that a strict symmetric monoidal category is a category $\mathcal{C}$ together with a bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and some natural transformations that serve to whitness the associativity, commutativity and unitality of the bifunctor, in the case of strict, up to natural isomorphisms. Monoidal categories are exactly the structures that are needed to define a monoid in your category, but they are also the monoid objects in the monoidal category of categories.

Example.
- Set *under cartesian product, unit is the singleton.*

- Top *under either cartesian product or smash product.*
- Spectra *under the smash product, with tensor unit* $\mathbb{S}$.
- mod_k *under tensor or direct sum with unit either* $\mathbb{Z}$ *or 0 respectively.*
- Ch_k *again under tensor or direct sum.*

1.2 Mays definition

Now consider a category Σ which may be called "the symmetric groupoid". Its objects are the finite (non-empty) sets $[n] = \{1, ..., n\}$ and morphisms are bijections,

$$\Sigma(n, m) = \begin{cases} \Sigma_n, & n = m \\ \emptyset, & \text{else.} \end{cases}$$

Σ-mod is the category $\text{Fun}(\Sigma^{op}, \mathcal{C})$ of Σ modules in $\mathcal{C}$ (which I have suppressed from the notation). Note that this is a category with natural transformations as the morphisms. Unpacking this definition we see that a Σ module in $\mathcal{C}$ is a sequence of objects $X_n \in \mathcal{C}$ along with the data of the maps

$$\Sigma_n = \Sigma(n, n) \to \text{Hom}_{\mathcal{C}}(X_n, X_n) = \text{End}(X_n).$$

Thus such a Σ module is a sequence of objects each with a Σ_n action.

Definition. *An operad in a strict symmetric monoidal category* $(\mathcal{C}, \otimes)$ *is a* Σ *module* P *with structure maps*

$$\gamma_{n,m_1,\dots,m_n} : P(n) \otimes P(m_1) \otimes \cdots \otimes P(m_n) \to P(\Sigma_i m_i), \qquad \forall n, m_i \geq 1$$

that further satisfies the conditions that the represent that the γ*'s are associative,* Σ *equivariant and unital; they interact sensibly with the monoidal structure.*

Operads form a subcategory of Σ-mod, with maps that make all the diagrams commute, that we call $\text{Op}_{\mathcal{C}}$ or just Op. If one drops all the equivariance conditions we get the definition of a non-Σ operad.

1.3 Examples

Associative. The category of sets is a monoidal category under cartesian product. Its tensor unit is any one point set, up to isomorphism [1]. There is an operad in Set given pointwise by the underlying set

$$\text{Ass}(n) = A(n) = \Sigma_n$$

where by convention $\Sigma_0, \Sigma_1 = [1]$. This is the 'associative' operad. Given a monoidal functor $\text{Set}, \times \to \mathcal{C}, \otimes$ we can compose with our operad to get an operad in $\mathcal{C}$. This is particularly useful because for most concrete categories there are such monidal functors, in the form of free functors. Two imporant examples are the functor into $\text{mod}_{\mathbb{Z}}$ give by $\mathbb{Z}[-]$ and the functor into Top given by the discrete topology. Thus we get notions of associative operads in modules and spaces. The Σ_n actions are clear and composition is 'block composition'.

Commutative. The category of symmetric monoidal categories with monoidal functors has an initial object. That initial object is just the category with one object and one map (the identity) with the trivial monoidal structure. This category admits a unique operad, namely the constant operad on the tensor unit and trivial maps. This operad is called Com or C. Since this category is initial in symmetric monoidal categories we can play the same push forward game as before but this time uniquely and moreover into any symmetric monoidal category.

Endomorphism. Consider a symmetric monoidal $(\mathcal{C}, \otimes)$. If $\mathcal{C}$ futher has an internal hom we denote it $\underline{\text{hom}}$. In this case there is a functor

$$\mathcal{E}nd_X(n) \quad := \quad \underline{\text{hom}}(X^{\otimes n}, X)$$

which becomes an operad with the structure maps given by

$$(\alpha; \beta_1, ..., \beta_n) \mapsto \alpha \circ (\beta_1 \otimes \cdots \otimes \beta_n).$$

This has a Σ_n operation that is induced from the action on $X^{\otimes n}$ by permuting the factors.

2 Algebras

If we have a morphism of operads $P \to \mathcal{E}nd_X$ then we call the object X a P-algebra. A morphism of P algebras $X, Y \in \mathcal{C}$ is a morphism in $\mathcal{C}$ that commutes with the operad action. The subcategory of $\mathcal{C}$ defined by P algebras is called P-alg.

Lemma. *A* Com *algebra in* $\text{mod}_{\mathbb{Z}}$ *is exactly a commutative algebra.*

Proof. Here we want to show an equivalence of categories. Do/.

Consider $X \in \text{Com} - \text{alg}$. Then we have a morphism of operads $\text{Com} \implies \mathcal{E}nd_X$. To make X a commutative, unital, associative $\mathbb{Z}$-algebra we will need to first define a $\mathbb{Z}$ bilinear pairing on X. Evaluating the natural transformation 'at the second level' gives a map

$$\text{Com}(2) = \mathbb{Z} \to \text{Hom}(X^2, X) = \mathcal{E}nd_X(2)$$

which we can further evaluate on 1 to get a pairing $X^2 \to X$. Since this map is internal to $\text{mod}_{\mathbb{Z}}$ it is already $\mathbb{Z}$ bilinear.

To get a unit we evaluate on 0, giving a map $\mathbb{Z} \to \text{Hom}(\mathbb{Z}, X)$, where $X^0 = \mathbb{Z}$. Again evaluating on 1 gives us a map $\mathbb{Z} \to X$ which we can take to be our unit. Is it compatible with the multiplication?

Associativity blah....

Finally to check commutativity we will use the equivariance. Using naturality with the maps in Σ we get that the moprphisms in Σ-mod are Σ_n quivariant on level n. Hence the morphism of operads is also equivariant, in particular we have equivariance of the map

$$\mathbb{Z} \to \text{Hom}(X^2, X)$$

under the Σ_2 actions. Since it acts trivially on the left and acts by permuting the inputs on the right, we see that permuting the inputs must have no effect and therefore the operation will be commutative!

Its more or less clear that this gives a full equivalence of categories as commuting with the operad structure will require the units to go to units, multiplication to be preserved etc.

□

This proof only seems to use the first like maybe 4 levels of the operad, can I truncate it and get the same thing... No because say we composed several of the level two guys they land in level 4 and above, then we want that to be 1 not 0. Obviously this comes from the fact that modules are themselves homotopically truncated.

Lemma. *A* Ass *algebra in* $\text{mod}_{\mathbb{Z}}$ *is exactly an associative algebra.*

What we see here is that being an *algebra* over these operads endows the object with a multiplication, one that is commutative or associative respectively, thus an *algebra* over these operads are *monoids*. Thus arises the conflation between algebras and monoids, comming from the change in perspective from the object to the map which gives the object that structure. This is familiar from a one categorical ring theory perspective on algebras, where an R-algebra is just another ring S equipped with a map $R \to S$ (scalar multiplication is given by pushing along this map).

3 E_∞ operads.

To define an E_∞ operad in a symmetric monoidal category we will further require that the category has a model structure. Following [BM03, Thm 3.1,3.2] under some weak hypotheses the category of operads in a symmetric monoidal *model* category can be given a model structure such that a map of operads is a weak equivalence iff the maps for each n level wise are given by weak equivalences in the target model category.

Now [BM03] define an operad to be Σ-cofibrant if the underlying sequence is cofibrant. For presumably technical reasons and using [BM03, Cor 4.5], if we have an operad P then the category of *homotopy* P-algebras is the category of P' algebras for some P' a cofibrant replancement of P, this is well defined up to Quillen equivalence. By [BM03, Prop 4.3] every cofibrant operad will be Σ-cofibrant.

Finally an E_∞ operad is such a cofibrant replancement of the canonical Com operad in the symmetric monoidal modal category. Thus an E_∞ algebra is an algebra over some cofibrant replacement of the constant functor on the tensor unit.

Example (Top)**.** *In* Top *with the standard (Serre fibration) modal structure the fibrant objects are everything and the cofibrant objects are (retracts of) CW complexes. Cofibrations are maps which have the LLP with respect to acyclic fibrations (i.e. with respect to each map which is both a Serre fibration and a weak homotopy equivalence).*

We want to replace the constant functor $*$ with something cofibrant. Because such a thing will have to be homotopy equivalent to the point level wise we know that it is level wise contractable. Less obvious is that to be cofibrant will requive the action to be free. Prove it. Thus the E_∞ operad in Top *is the functor that assigns level wise a Σ_n free space that is contractable and a CW complex. Canonically this is $E\Sigma_n$ (to make this properly into an operad we need the structure maps, see the Barratt-Eccles operad).*

Example (Chain complexes)**.** *Here the model structure is chain homotopy and level wise surjections. Cofibrations are monomorphisms with projective kernels. We need to asign something that is level wise weakly equivalent to the constant chain of $\mathbb{Z}$ with all isomorphisms. That is some acyclic chain. Again the action should be free.*

Remark. In [BM03] the model structure they define has weak equivalences and *fibrations* defined pointwise. This is why here we take a *cofibrant* replacement, as all the non-trivial group actions have been forced into whatever the class of cofibrations becomes under the pointwise specicications of fibrations and weak equivlances.

3.1 Linear isometries operad

There are many models of the E_∞ operad in Top. I will mention there names; little disk, little cubes, configuration spaces and the linear isometries operad. The linear isometries seems to me a nice concrete model to have in mind, the others are more suitable for E_{n} algebras and then one must compute a limit in the category of operads to get E_∞ out of them.

First we have the symmetric monoidal category **LI**, with objects real inner product spaces of countable dimension and linear isometric embeddings, the monoidal structure is given by direct sum. Objects of **LI** are topologised by metric topology for finite dimensional spaces and by the direct limit topology over all inclusions of subspaces for infinite dimensional ones. Morphisms are topologised using compactly generated function space topology.

We denote $\mathbb{R}^\infty \;\; := \;\; \lim_{\rightarrow} \mathbb{R}^n$. The operad $\mathcal{L}i$ is defined pointwise as

$$\mathcal{L}i(n) \;\; := \;\; \mathbf{LI}(\oplus^n \mathbb{R}^\infty, \mathbb{R}^\infty).$$

The symmetric group action is by permuting the inputs and the structure maps are given by composition of the direct sum of the maps.

Lemma. *For all $V \in \mathbf{LI}$ we have that*

$$\mathbf{LI}(V, \mathbb{R}^\infty) \simeq *,$$

the space is contractable.

Proof. Break $\mathbb{R}^\infty$ into an even and odd graded peices, show that there is a homotopy from the whole space to the even half. Likewise for any embedded space we can break it into even and odd parts and produce a homotopy between them.

From this space level homotopy we can construct a homotopy from the identity of $\mathbb{R}^\infty$ to the inclusion of $\mathbb{R}^\infty$ into the even half of $\mathbb{R}^\infty$. Note that we want to perform these homomtoies 'through isometries' since those are the elements of the Hom sets. This is done by *rotating* vectors, that is the homotopies are rotations, as opposed to linear interpellation.

Then for any isometry we can put its image in the even graded part using the above homotopy (post compose with it). Now we can construct a homotopy between any map that lies entirely in the even part (up to homotopy any map) and any map that lies in the odd part as they are orthogonal and we can therefore apply rotations.

□

Note in particular that linear isometries from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ are also contractable.

Lemma. *The action of the symmetric group on $\mathbf{LI}(\oplus^n \mathbb{R}^\infty, \mathbb{R}^\infty)$ is free.*

Proof. Sufficient to see it for the case of $n = 2$. We want to show that every $f \in \mathbf{LI}(\oplus^2 \mathbb{R}^\infty, \mathbb{R}^\infty)$ has a trivial stabiliser. Let $\sigma \in \Sigma_2$ be the non-trivial element. Assume for a contradiction that

$$\begin{aligned} \sigma f &= f \\ \implies \forall x, y \quad f(x, y) &= f(y, x) \\ \implies f(x, y) - f(y, x) &= 0 \\ \implies (x - y, y - x) &\in \ker f \end{aligned}$$

But since f is an embedding we see that $\ker f = 0$ hence $x - y = y - x = x = y = 0$ a contradiction.

□

Hence by our characterisation of E_∞ in Top we get that $\mathcal{L}i$ is an E_∞ operad.

Remark. We havent talked about E_n algebras. They are for commutativity only up to n-homotopies. To define them abstractly is much less simple than for E_∞ which is entirely canonical. To define them concretely use the little disk operad

3.2 Recognition

May originally invented operads to prove the following statement,

Lemma (Recognition)**.** *A connected space is an E_∞ algebra iff it is of the homotopy type of an infinite loop space.*

Using $\mathcal{L}i$ we can get easy proofs that certain spaces are infinite loops spaces. For instance consider a (continuous monoidal) functor $F : \mathbf{LI} \to \text{Top}$. Then we can apply this functor on hom sets (spaces)

$$\mathbf{LI}(\oplus^n \mathbb{R}^\infty, \mathbb{R}^\infty) \to \text{Hom}(F(\mathbb{R}^\infty)^n, F(\mathbb{R}^\infty))$$

which will form a map of operads $\mathcal{L}i \implies \mathcal{E}nd_{F(\mathbb{R}^\infty)}$. Thus $F(\mathbb{R}^\infty)$ is a $\mathcal{L}i$ algebra and therefore an infinite loop space.

For example a functor of the form F we can consider $V \mapsto O(V)$, or $V \mapsto BO(V)$. Thus we have shown that O and BO are infinite loop spaces. The same goes for U, Sp and their classifying spaces, by tensoring with either $\mathbb{C}$ or $\mathbb{H}$.

Remark. (Waffle) So what have we seen. We have seen that there is an operad that classifies commutative monoid structures. We have seen that when we make that operad homotopically sensible (by cofibrantly replacing it) we get what is called the E_∞ operad. We have seen that algebras over this operad are exactly infinite loop spaces.

The moral of the homotopic replacement is that we want commutativity only up to homotopy. Because we have replaced the condition of 'commuting up to homotopy' with the structure of an operad we have to actually pick the concrete homotopies. This is what is meant by homotopy commutative. It is a monoid such that the operation commutes *up to some specified system of related homotopies that are related associatively etc.* To make this precise you will find yourself exactly writing down the E_∞ operad algebra definition.

Thus in this precise sense infinite loop spaces are the homotopically coherent commutative groups. Tying this into our previous discussion of spectra we can see that Ω spectra are therefore homtopy coherent *abelian* groups.

3.3 Free Resloution Operad

Later it will be nice to see explicitly a construction of an E_∞ algebra structure in chain complexes, for this we will obviously need an explicit model of the operad. We will follow [Smi82, Ex 10]. The process is inductive and free, in the sense that we will apply two free functors

$$\mathrm{Fun}(\Sigma^{\mathrm{op}}, \mathrm{Ch}) \xrightarrow{T} \text{non } \Sigma \text{ Op} \xrightarrow{\Sigma} \mathrm{Op}.$$

We will construct 'skeleta' $M^{(n)}$ and set $E^{(n)} = T\Sigma M^{(n)}$.

Base Case. We start with the Com operad, so let $M^{(0)} = \mathrm{Com}$. Let $E^{(0)} = T\Sigma M^{(0)}$ and we will also define a differential $\partial^{(0)} = 0$.

Induction. Assume that we have constructed $M^{(n)}$ and $E^{(n)}$ as well as a differential $\partial^{(n)}(n)$: $E^{(n)}(n) \to E^{(n)}(n-1)$. Now define

$$M^{(n+1)}(i) = \begin{cases} M^{(n)}(i), & i \neq n+1 \\ \ker\left(\partial^{(n)}(n) : E^{(n)}(n) \to E^{(n)}(n-1)\right), & i = n+1. \end{cases}$$

As mentioned we then set $E^{(n+1)} = T\Sigma M^{(n+1)}$. Finally we define the differential

$$\partial^{(n+1)}(x) = \begin{cases} \partial^{(n)}(x), & x \in E^{(n+1)}(i), i \neq n+1 \\ T\Sigma\iota(x), & x \in E^{(n+1)}(n+1) \end{cases}$$

where $\iota : M^{(n+1)}(n+1) \to E^{(n)}(n)$ is the inclusion, which we can extend linearly over all of E.

Concluding. Then by definition $E^{(n)}$ is Σ free up to dimension $n-1$ and by the classic Steenrod construction is also acyclic! Hence as $n \to \infty$ we get an E_∞ operad.

4 E_∞ algebras give operations on homology.

Lemma (Thm 7.2 [KM18])**.** *Consider the symmetric monoidal category* $(\mathrm{Ch}_{\mathbb{Z}}, \otimes)$ *of chain complexes. Let* $A \in \mathrm{Ch}_{\mathbb{Z}}$ *be an* E_∞ *algebra. Then for* $s \geq 0$ *and* p *a prime there exist natural homology operations,*

$$Q^s : H_q(A; \mathbb{Z}_p) \to \begin{cases} H_{q+s}(A; \mathbb{Z}_p), & p = 2 \\ H_{q+2s(p-1)}(A; \mathbb{Z}_p), & p \neq 2 \end{cases}$$

satisfying 6 conditions,

1. *$Q^s = 0$ if $p = 2, s < q$ or $p > 2, 2s < q$*
2. *$Q^s(x) = x^p$ if $p = 2, s = q$ or $p > 2, 2s = q$*
3. *$Q^s(1) = 0$ for $s > 0$ where $1 \in H_0(A; \mathbb{Z}_p)$ is the identity (of the abelian group).*
4. *(Cartan) $Q^s(xy) = \sum Q^t(x)Q^{s-t}(y)$*
5. *Adem relations*
6. *Relations with the Bockstein.*

The reference cites a proof in [Coh76, I.§1.Thm 1.1] for when you start with a space which is an E_∞ operad in which the chain complex is then given an E_∞ structure, and says that the general case is more or less the same. We will try to work through the general case.

Defining the operation. So assume that A is an E_∞ algebra in chain complexes. Thus there is a Σ_p equivariant map, which induces an equivariant map on the tensor,

$$\xi : E(p) \to A^{\otimes p} \implies \bar{\xi} : E(p) \otimes \mathbb{Z}_p \to (A \otimes \mathbb{Z}_p)^{\otimes p}$$

where E is the E_∞ operad. Note that $E(p)$ is a Σ_p free resolution of $\mathbb{Z}$ and hence the tensor is a Σ_p free resolution of $\mathbb{Z}_p$This seems to hold for any ring, not special for the finite fields.... Note that $\mathbb{Z}_p \subseteq \Sigma_p$ and hence (by homological algebra) $E(p) \otimes \mathbb{Z}_p$ is also a $\mathbb{Z}_p$ free resolution of $\mathbb{Z}_p$. There is another more canonical $\mathbb{Z}_p$ free resolution of itself, with a single generator in each degree, that we denote W. By the 'fundamental theorem of homological algebra' we can extend the identity $H_0(W) \to H_0(E(p) \otimes \mathbb{Z}_p)$ to a quasi-isomorphism of chain complexes $j : W \to E(p) \otimes \mathbb{Z}_p$ (as both are a-cyclic free resolutions of the same thing over the same thing). Thus we have a map

$$\theta : W \otimes (A \otimes \mathbb{Z}_p)^{\otimes p} \xrightarrow{j \otimes \mathrm{id}} (E(p) \otimes \mathbb{Z}_p) \otimes (A \otimes \mathbb{Z}_p)^{\otimes p} \xrightarrow{\bar{\xi}} A \otimes \mathbb{Z}_p.$$

This induces a map on homology θ_*. Denoting the generator of W_i as e_i and letting $x \in H_q(A; \mathbb{Z}_p)$ we make the definitions:

- $Q_i(x) = \theta_*(e_i \otimes x^p)$, note that there are some isomorphisms applied here to move cohomology of A to the outside.
- For the prime $p = 2$,
$$Q^s(x) = \begin{cases} 0, & s < q \\ Q_{s-q}(x), & s \geq q. \end{cases}$$
- For the primes $p \geq 3$,
$$Q^s(x) = \begin{cases} 0, & 2s < q \\ (-1)^{s + \frac{q(q-1)(p-1)}{4}} ((\frac{1}{2}(p-1))!)^q Q_{(2s-q)(p-1)}(x), & 2s \geq q \end{cases}$$

Checking some of the properties. Property (1) is immediate from the definition. Property (2) follows from some thinking: Say for $p = 2$ we want the $s = q$ map and so we want Q_0. In this case we want to compute the action of e_0 on x^p, which is given by the action of the generator of $E(p)_0$, *on homology* this is exactly the action of the operad $\mathrm{Com}(p)$ which is trivial (it is given by this by definition of E above), thus we see that we have simply the map $(A \otimes \mathbb{Z}_p)^{\otimes p} \to A \otimes \mathbb{Z}_p$ that is the unit in $\mathrm{Hom}((A \otimes \mathbb{Z}_p)^{\otimes p}, A \otimes \mathbb{Z}_p)$. This map is by definition "x^p", it is making the module $H_*(A; \mathbb{Z}_p)$ an operad over Com, so a commutative algebra, when this is the chain complex of a space then we see that it will be the cup product etc.

Intuitively property (3) should hold because the unit is already commutative on the nose. Strictly recall how the operad will act on the identity in one of the slots, by basically ignoring it (sending to the tensor unit). Hence the map will factor through the degree zero action on $E(0) \to \mathcal{E}nd_A(0) \cong A$. Because e_i is in degree $i > 0$ we see that in $E(0)$ it is zero on homology since $E(0)$ is acyclic.Ok there is some reworking necissary. Via these notes, we have the notion of $E(0)$ etc which was previously not defined. This is a "unital" operad, and comes with degeneracy maps. Here then the point is then that we can degenerate e_i to make sense of how the action factors through the level 0 case.

Per AI: The Cartan formula comes from Hopf algebra structure on E and W, where the comultiplication on W is exactly the Cartan formula, this permeates to give the same structure on the induced operations.

Remark. Notice that it does not follow here that the operations are stable and we will see examples that are manifestly unstable (Dyer-Lashoff).

Remark. To get he $\mathbb{Z}_p$ homology of an integral chain we just first tensor the whole chain with $\mathbb{Z}_p$.

Remark. Some important things to notice are the following. First we can see that at least from this definition *these* homology operations (there may be others...) are in bijection with and determined by the e_i, thus in some sense we can identify the e_i as 'the homology operations'. Given this we see that the e_i are a priori the generators of W_i but a posteriori generators of the *group* homology $H_*(\mathbb{Z}_p; \mathbb{Z}_p)$, which is given by *tensoring over* $\mathbb{Z}_p[\mathbb{Z}_p]$! One $\mathbb{Z}_p$ is the coefficient ring, the other the subgroup of Σ_p. Now notice that

$$H_*(\Sigma_p; \mathbb{Z}_p) \cong H_*\big((E(p) \otimes \mathbb{Z}_p) \otimes_{\mathbb{Z}_p[\Sigma_p]} \mathbb{Z}_p\big).$$

Recall that $(E(p) \otimes \mathbb{Z}_p) \otimes_{\mathbb{Z}_p[\Sigma_p]} \mathbb{Z}_p \cong (E(p) \otimes \mathbb{Z}_p)/\Sigma_p$. Our j_* map therefore induces a map

$$H_*(\mathbb{Z}_p; \mathbb{Z}_p) \to H_*(\Sigma_p; \mathbb{Z}_p).$$

Therefore the operations are exactly elements in $H_*(\Sigma_p; \mathbb{Z}_p)$ given by the image of $j(e_i)$. Now the fact is that this j_* is an isomorphism for $p = 2$ and always a surjection. The indices in the $p > 2$ case are exactly parametrising its non-zero image, the other Q_i without a factor of $p - 1$ would be zero. I need to replace the subgroup with a different notation!!!!!!!!!!

5 Examples of E_∞ algebras

5.1 Cochain of a space.

Following [Smi82, §2 Thm 1], also proved in [BF04].

We will describe an E_∞ *co*algebra structure on the *chain complex* of any space. Dualising this provides the required algebra structure on the cochain complex (with field coefficients).

So let X be the given space, let E be the model of the E_∞ operad (pointwise some acyclic and free chain complexs) and let $C_*(X; \mathbb{Z}_p)$ be its singular chain complex tensored with $\mathbb{Z}_p$. Then recall that a coalgebra structure is a map of operads

$$E \to \mathcal{E}nd^{C_*(X;\mathbb{Z}_p)}$$

Where $\mathcal{E}nd^{C_*(X;\mathbb{Z}_p)}(n) := \mathrm{Hom}(C_*(X; \mathbb{Z}_p), \otimes^n C_*(X; \mathbb{Z}_p))$. That means that we need to contruct maps of chain complexes $E(n) \to \mathrm{Hom}(C_*(X; \mathbb{Z}_p), \otimes^n C_*(X; \mathbb{Z}_p))$.

Base case of induction. There are natural maps $\varphi_j \in \mathrm{Hom}(C_*(X; \mathbb{Z}_p), \otimes^j C_*(X; \mathbb{Z}_p))$ preserving augmentation What are they?????(the map in degree zero to the coefficients) that are unique up to homotopy. This forms a map $M^{(0)} \to \mathcal{E}nd^{C_*(X;\mathbb{Z}_p)}$, sending the unique generator in each degree to the given maps.

Inductive step. We can use the acyclic model theorem (similar but more powerful than the acyclic carrier theorem, see here) to extend the chain maps from $M^{(n)}$ to $M^{(n+1)}$.

The homology operations induced from this complex for the groups $\mathbb{Z}_p$ are the power operations. Moreover they assemble into natural transformations on the whole category of Top.

5.2 Chain complex of an E_∞ algebra

Example (Chain complex of an E_∞ algebra, [KM18] I.§5)**.** *We have already noted that given a monoidal functor we can push forward operads along them to move the operads to different categories. Given a morphism of operads we can also do the same thing, by composing on both sides. This is well defined by functoriality.*

Given an E_∞ algebra in Top *we can apply the singular chain complex functor to get an algebra in simplicial sets. Applying the free $\mathbb{Z}$ module functor gives an algebra in simplicial ungraded $\mathbb{Z}$ modules. Apply Dold-Kan to get a chain complex. This is intuitively clear however there are technicalities here. We need all the functors to preserve the symmetric monoidal structure, which can be hard to check. Moreover one must check that the operad we are an algebra over is E_∞ in both cases. Note that the monoidality in this setting is mostly just the Eilengberg-Zilber theorem!*

The induced homology operations are the Dyer-Lashoff operators, since May gives us that the operations are natural in the chain complexes and the chain complexes are themselves natural in the spaces we see that we get natural transformations on homology functors, where we restrict the domain to the subcategory of Top given by infinite loop spaces and infinite loop space maps.

Note that these operations are not stable, the suspension of a loop space is not necissarily a loop space in a compatable way (consider the James construction).

Remark. From this perspective we can see that the Steenrod squares are exactly the homotopies that make the normal cup square coherently commutative. In general we can see that directly from the formula for the i-th cup product: ....Fillll

Remark. They are algebras over other rings too but the operations dont exist I think

6 Appendix: Diagrams

1. Associativity. Given natural numbers m_i for $1 \leq i \leq n$ and $l_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, define

$$m := \sum_{1 \leq i \leq n} m_i, \ l_i := \sum_{1 \leq j \leq m_i} l_{i,j}, \ l := \sum_{1 \leq i \leq n} l_i$$

and

$$\mathbf{m} := (m_1, \dots, m_n), \ \mathrm{l} := (l_{1,1}, \dots, l_{n,m_n}),$$
$$\mathrm{l}_i := (l_{i,1}, \dots, l_{i,m_i}), \ \mathrm{l}' := (l_1, \dots, l_n).$$

Let

$$\mathcal{P}[\mathbf{m}] := \mathcal{P}(m_1) \odot \cdots \odot \mathcal{P}(m_n), \quad \mathcal{P}[\mathrm{l}_i] := \mathcal{P}(l_{i,1}) \odot \cdots \odot \mathcal{P}(l_{i,m_i}),$$
$$\mathcal{P}[\mathrm{l}] := \mathcal{P}(l_{1,1}) \odot \cdots \odot \mathcal{P}(l_{n,m_n}) = \mathcal{P}(\mathrm{l}_1) \odot \cdots \odot \mathcal{P}(\mathrm{l}_n)$$

and

$$\mathcal{P}[\mathrm{l}'] := \mathcal{P}(l_1) \odot \cdots \odot \mathcal{P}(l_n).$$

Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}(n) \odot \mathcal{P}[\mathbf{m}] \odot \mathcal{P}[\mathrm{l}] & \xrightarrow{\rho} & \mathcal{P}(n) \odot (\mathcal{P}(m_1) \odot \mathcal{P}[\mathrm{l}_1]) \odot \cdots \odot (\mathcal{P}(m_n) \odot \mathcal{P}[\mathrm{l}_n]) \\ & & \downarrow \mathbb{1} \odot \bigodot_{i=1}^{n} \gamma_{m_i;\mathrm{l}_i} \\ \downarrow \gamma_{n,\mathbf{m}} \odot \mathbb{1} & & \mathcal{P}(n) \odot \mathcal{P}[\mathrm{l}'] \\ & & \downarrow \gamma_{n;\mathrm{l}'} \\ \mathcal{P}(m) \odot \mathcal{P}[\mathrm{l}] & \xrightarrow{\gamma_{m;\mathrm{l}}} & \mathcal{P}(l) \end{array} \tag{1.4}$$

where ρ applies the symmetry in $\mathcal{C}$ to permute the factors in $\mathcal{P}[\mathbf{m}] \odot \mathcal{P}[\mathrm{l}]$ to give a product of the factors $(\mathcal{P}(m_i) \odot \mathcal{P}[\mathrm{l}_i])$ for $1 \leq i \leq n$.

2. Equivariance. Given $\sigma \in \Sigma_n$ and an n-tuple $\mathbf{m} = (m_1, \dots, m_n)$, define $\sigma\mathbf{m} := (m_{\sigma^{-1}(1)}, \dots, m_{\sigma^{-1}(n)})$. Let $\overline{\sigma} : \mathcal{P}[\mathbf{m}] \to \mathcal{P}[\sigma\mathbf{m}]$ be the permutation of the factors given by the symmetry in $\mathcal{C}$ and $\sigma_{\mathbf{m}}$ the block permutation described in Definition 1.2. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(n) \odot \mathcal{P}[\mathbf{m}] & \xrightarrow{\mathbb{1} \odot \overline{\sigma}} & \mathcal{P}(n) \odot \mathcal{P}[\sigma\mathbf{m}] \\ & & \downarrow \gamma_{n;\sigma\mathbf{m}} \\ \downarrow \sigma \odot \mathbb{1} & & \mathcal{P}(m) \\ & & \downarrow \sigma_{\mathbf{m}} \\ \mathcal{P}(n) \odot \mathcal{P}[\mathbf{m}] & \xrightarrow{\gamma_{n;\mathbf{m}}} & \mathcal{P}(m). \end{array} \tag{1.5}$$

3. Unit. If $\mathbf{1}$ is the unit object of $\mathcal{C}$, then there is an $\eta : \mathbf{1} \to \mathcal{P}(1)$ such that the composite morphisms

$$\mathcal{P}(n) \odot \mathbf{1}^{\odot n} \xrightarrow{\mathbb{1} \odot \eta^{\odot n}} \mathcal{P}(n) \odot \mathcal{P}(1) \odot \cdots \odot \mathcal{P}(1) \xrightarrow{\gamma_{n,1,\ \ ,1}} \mathcal{P}(n)$$

and

$$\mathbf{1} \odot \mathcal{P}(m) \xrightarrow{\eta \odot \mathbb{1}} \mathcal{P}(1) \odot \mathcal{P}(m) \xrightarrow{\gamma_{1,m}} \mathcal{P}(m)$$

are respectively the iterated right unit morphism and the left unit morphism for the underlying monoidal category $\mathcal{C}$.

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